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Reputation in Multi-unit Ascending Auction with Common Values

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Abstract

This paper considers a multi-unit ascending auction with two players and common values. A large set of equilibria in this model is not robust to a small reputational perturbation. In particular, if there is a positive probability that there is a type who always demands many units, regardless of price, then the model has a unique equilibrium payoff profile. If this uncertainty is only on one side, then the player who is known to be normal lowers her demand in order to stop the auction immediately at the reserve price. Hence, her possibly committed opponent buys all the units she demands at the lowest possible price. If the reputation is on both sides, then a War of Attrition emerges.

Keywords: Multi-unit auction, uniform price, ascending auction, reputation, aggressive bidding.

JEL Classification: D44

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1 Introduction

The simultaneous ascending auction is a remarkable application of economic theory to the design of real economic mechanisms. Roughly speaking, in such an auction the bidding continues until the total demand decreases so that it matches the supply. This format has been used by the US Federal Communication Commission since 1994 for the purpose of allocating radio spectrum for telecommunication services. More recently, its version was used in auctioning off UMTS licenses in Germany and Austria.

Despite a great progress in understanding various incentives faced by bidders in this auction, the simplest theoretical model suffers from a serious deficiency: there is a multiplicity of equilibria, and consequently the model has virtually no predictive power.

This paper shows, that this multiplicity of equilibria is not robust. Introduce an arbitrarily small but positive probability that bidders may be of a behavioral type that demands large quantities, even if prices are high. Then there is a unique equilibrium payoff profile and that fact does not depend on the size of the perturbation. This is referred to as reputational perturbation, because it creates incentives for normal bidders to maintain reputation, namely to mimic those who are programmed to demand large quantities. This result is obtained in a model of ascending auction with two bidders and many units of a homogeneous good for sale, in which the value of a unit is the same for both bidders regardless of quantities acquired, but it is not necessarily common knowledge.

If the reputation is one-sided (only one player may be behavioral, but the other one is certain to be normal), then in any equilibrium there is no delay. The player who has no reputation decreases his demand to the level that clears the market, in order to prevent the price from rising. If such a strong asymmetry occurs, the seller will not obtain any additional revenue beyond the reserve price.

Two-sided reputation is investigated by means of a simpler model with known common values. The game has a flavor of a War of Attrition; its unique equilibrium is in mixed strategies. Each normal bidder decreases his demand to clear the market with some density over some price interval. However, some surplus may stay with the bidders, since the expected price may still be less than the value of the object.

The overall message for the seller interested in maximizing the revenue is negative, though, even in the case of two-sided reputation. The revenue may depend sensitively on the value of parameters that are difficult to observe, such as the relative likelihood of behavioral types.

2 Literature and contributions of this paper

It is known that there is a great multiplicity of equilibria in a standard ascending or uniform price multi-unit auction, particularly if the values are common. Very low prices, even equal to an exogenous reserve price, may be supported by an equilibrium. For instance, assume that bidders implicitly split the total number of units among themselves and stick to their designated shares for any price. No player has any incentive to deviate from this strategy profile. Any deviation to a lower demand does not affect the price, but assigns fewer units to the player and hence results in lower payoff. Any deviation to a higher demand rises the price but ultimately does not affect the assignment, again resulting in lower payoff. Equilibrium strategy profile results in obtaining the agreed share at the lowest possible price. Other allocations and prices can be supported as equilibrium outcomes too.

Obviously, this collusive-seeming equilibrium is not revenue-maximizing. A possibility of low revenues was first noticed by Wilson (1972), in the context of a divisible-object and sealed-bid version of the model, see also Back and Zender (1993). Riedel and Wolfstetter (2006) show that if the marginal values are known, decreasing in quantity and not equal across agents, then there is a unique equilibrium through iterated elimination of dominated strategies, in which the division of objects is efficient and the price is minimal. This result does not cover the case of common values; this is this gap that this paper aims to fill.

Some recommendations were given to the seller who, for some reasons, must use uniform price mechanism. Back and Zender (1999) and McAdams

(2007) give the seller an opportunity to adjust the supply strategically after bidders announced their demands. Since bidders' collusive behavior is sensitive to this quantity, they are not able to coordinate so well.

Another approach assumes that there are some noisy players who purchase some part of the total supply, leaving an uncertain residual for the strategic bidders. The probability of such a supply variation can be arbitrarily small, and still a unique equilibrium is selected as long as this noise has a full support. This method of equilibrium selection was introduced by Klemperer and Meyer (1989); Rostek et al (2009) is a recent application. The model in this paper introduces a perturbation too, but of a different type.

All of the above papers consider multi-unit auctions in a sealed-bid context. In contrast to that, this project focuses on ascending version of the model, similar to Milgrom (2000) or Cramton (2006). This rises a possibility of interesting dynamics, such as collusion (for instance, Cramton and Schwartz (2002)). This paper studies dynamic reputational effects.

From the point of view of the reputation literature the model of this paper is closely related to Myerson (1991, section 8.8) and Abreu and Gul (2000), who analyze bargaining problem with reputation. A multi-unit ascending auction can be seen as a mechanism in which bidders negotiate/bargain how to split the objects among themselves. One difference is how the cost of bargaining is specified – in the classical bargaining model this is discounting, in the multi-unit ascending auction this is through increasing price. The connection between bargaining and ascending multi-unit auctions was made by Ausubel and Schwartz (1999)

An important contribution of this study is an extension of Myerson's (1991) result to the case in which common values are not commonly known, in the context of an auction. In particular, reputational effects completely dominate what players may know about the value of the objects. For instance, suppose that player one knows the value and player two does not; standard logic of the winner's curse suggests that player two should yield early. However, if player two has a reputation as defined below (no matter how small) and player one does not, it is player one who would yield immediately.

Myatt (2005) studies a War of Attrition with ex ante asymmetry in values rather than with common values; his result is that the player with lower private value exits almost immediately if the two-sided reputational perturbation is small enough. To complement this result, this paper provides an example with one-sided reputation, like Myerson (1991), and demonstrates that there may be multiplicity of equilibria if values are ex ante asymmetric, like in Myatt (2005).

3 Model

There are $L \geq 3$ units of a good for sale. Let $r \geq 0$ be an exogenous reserve price. There are 2 bidders, each one can buy at most $L-1$ units.

The format of an auction is ascending. The price is continuously rising and each bidder publicly announces how many units he wants to buy at a current price. Bidders announce their individual demands by means of a dial that can be rotated only towards lower values. Assume that bidders can decrease their demand only by one unit at a time. Bidders can react instantaneously to demand reductions by the opponent. If two bidders decide to decrease demand at the same price, the nature uses a fair coin to determine whose reduction is announced. The auction stops at the market clearing price, the first price where the excess demand is zero.

The information is asymmetric in two ways. At the beginning of the auction, nature chooses for each bidder $i = 1, 2$ (i) a type $t_i \in \{0, x_i\}$, where $t_i = 0$ denotes a "normal" type and $t_i = x_i$ denotes a "behavioral" type; and (ii) a signal $\theta_i \in \Theta_i$. For simplicity assume that the set of signal profiles $\Theta = \Theta_1 \times \Theta_2$ is finite; let $\theta = (\theta_1, \theta_2)$. Bidder i learns his type and his signal, (t_i, θ_i). Assume that θ, t_1 and t_2 are independent, but θ comes from an unrestricted joint probability distribution, where q_θ is the probability of θ. In the section discussing two-sided reputation, it will be assumed that Θ is a singleton.

With probability $1 - \mu_i$ bidder i is a normal type, $t_i = 0$. The gross value of one unit when θ is realized is $v_\theta > 0$. This formulation makes sure that the value of the unit is constant across agents and does not depend on the quantity acquired by the bidder. This value however does depend on the profile of signals, and therefore is only partially known to bidder i, who knows the realization of signal θ_i but not of θ_j. Normal type of i has the realized net payoff equal to $(v_\theta - p) z_i$, where p is the final price and z_i is the final allocation of units to player i.

With probability $\mu_i \geq 0$ bidder i is a behavioral type, $t_i = x_i$. Behavioral type always demands x_i units for any price, a quantity that is treated as exogenous in this paper, sometimes referred to as "greediness" of i. To assure that seller's revenue is bounded, assume that the behavioral type reduces his demand from x_i to zero at price $\max_\theta v_\theta$.

History at price p must describe how and when the bidders decreased their individual demands prior to price p. Thus, divide the path into stages, where stage n also denotes the size of the excess demand at that stage. In other words, stages of the ascending auction are counted backwards, with stage 1 being the last one. Let the initial excess demand be $\bar{n}$. Consider an auction that reached a stage in which excess demand is $n \in \{\bar{n}, \bar{n}-1, ..., 1\}$; let p^n be the price at which the last demand reduction occurred and let z_i^n be the resulting demand of bidder i. The history in stage n, for any price $p \geq p^n$, will be identified with a list of prices, at which reduction of demand occurred, and the resulting demands:

$$h^n = \left\{ \left(p^{\bar{n}}, z_1^{\bar{n}}, z_2^{\bar{n}}\right), \left(p^{\bar{n}-1}, z_1^{\bar{n}-1}, z_2^{\bar{n}-1}\right), ..., \left(p^n, z_1^n, z_2^n\right) \right\}$$

where for every $n_o = \bar{n}-1, ..., n$ the following holds: $p^{n_o} \geq p^{n_o+1}$ and $z_i^{n_o} \in \left\{z_i^{n_o+1} - 1, z_i^{n_o+1}\right\}$ for both $i = 1, 2$ and $z_i^{n_o} = z_i^{n_o+1} - 1$ for some i. Each element $(p^{n_o}, z_1^{n_o}, z_2^{n_o})$ denotes an event that one of the bidders decreased his demand at price p^{n_o} by one unit. If $p^{n_o} = p^{n_o+1}$, then after one of the bidders decreased his demand at price p^{n_o+1}, someone decreased the demand again immediately. Initial element of this list denotes the reserve price $p^{\bar{n}} = r$ and $z_i^{\bar{n}} \leq L - 1$ is an initial demand of player i. The final element of this list must have $n > 0$ for the auction to still continue after p^n.

The multi-unit ascending auction described above will be regarded as a series of simultaneous-move auctions. Specifically, fix a history h^n. It means that the price and demands are (p^n, z_1^n, z_2^n). Assume that bidders play a simultaneous-move game, in which bidder $i = 1, 2$ chooses a "bid" $b_i \geq p^n$. If bidder 1 has a bid lower than bidder 2, then his bid becomes a new standing price in the next stage, $p^{n-1} = b_1 = \min\{b_1, b_2\}$, the new demand of bidder 1 is $z_1^{n-1} = z_1^n - 1$, and the demand of bidder 2 remains unchanged at $z_2^{n-1} = z_2^n$. If there is a tie, $b_1 = b_2$, then the player whose demand is to be reduced is determined randomly. The new history is $h^{n-1} = \left\{h^n, \left(p^{n-1}, z_1^{n-1}, z_2^{n-1}\right)\right\}$.

Let $F_i(b_i|\theta_i)$ be a probability that a normal bidder i who observed θ_i decreases his demand at or before price $b_i \geq p^n$, where the dependence on the history h^n is implicit. A strategy for the entire game is a collection of these functions, one for each history and a probability distribution over $\{0, 1, ..., L-1\}$ which picks an initial demand (if history is null).

4 Behavioral type on one side

This section assumes that bidder i is known to be normal, $\mu_i = 0$, but the probability that player j is of behavioral type is $\mu_j > 0$. Consider strategies in which normal bidder j starts demanding x_j at the reserve price and suppose that at the beginning of this ascending auction there is some excess demand, $n \geq 1$. Let $y_i = L - x_j$ be a residual quantity for bidder i, once the behavioral demand of bidder j is fully satisfied.

The following is the main result of this section. All proofs are in the Appendix.

Proposition 1 *Consider any stage $n \geq 1$ with initial price p^n, in which $\mu_j^n > 0$, $\mu_i^n = 0$. Then the only equilibrium market clearing price is p^n and allocation to bidder i is y_i units.*

This is a closely related to the result developed by Myerson (1991, section 8.8) in the context of bargaining. The construction measures two forces that would have to occur in any equilibrium in which both players do not reduce their demands immediately. The first force is that after player j reduces his demand, thus revealing his normality, player i must obtain particularly high payoff, call it w_i. This is because if j does not reduce his demand (which happens with positive probability), then player i obtains particularly low payoff from insisting on his high initial demand for some time and yielding at a higher price. Since the expected payoff of player i over these two events has to be at least as good as the available option of reducing the demand immediately, the promise of w_i must be attractive. The second force is that when player j reduces his demand, he must obtain high payoff, call it w_j, higher than the available option of waiting until player i ultimately reduces his demand. The proof shows that these requirements for high payoffs w_i and w_j are not simultaneously feasible.

Proposition 1 makes sure that the construction of Myerson (1991) for bargaining with discounting is still valid in the current auction context in which cost of delay is linear. More importantly, the argument sketched above works even if common values are not commonly known, as assumed in Myerson (1991). Two remarks about common values should be made.

The first interesting observation is that this result does not depend on set Θ or the distribution over the signals. For example, suppose that player i knows the true value of the unit, while player j does not. One may argue that this additional knowledge gives player i an advantage: it makes bidding less attractive to player j, due to winner's curse. This may force player j to yield before player i.

Equilibrium of this type is reported by Milgrom and Weber (1982) (see Theorem 7) in the context of common-value second-price auctions with a single unit. The fact that signal of j is a garbling of a signal of i implies that in any equilibrium it must be that one of the bidders wins with probability zero. Proposition 1 says that in the current context this consideration is completely dominated by a possible existence of a behavioral type on one side. Regardless of which player has a superior knowledge about the value of the object, the player who is known to be normal yields immediately against his possibly behavioral opponent.

Second remark is that the assumption about common values cannot be relaxed. Without it, there may be multiplicity of equilibria even if there is a reputation (on one side). The following example considers players whose values are asymmetric ex ante, a case studied by Myatt (2005), but focuses on one-sided reputation, like in Myerson (1991).

Example 1 *Suppose that* $r = 0$, $x_i = x_j = 2$, *and* $L = 3$. *Values are commonly known but they are not common. In particular, assume that* i *is value-strong, but reputation-weak. That is, let* $v_j < v_i$ *and assume that player* i *is known to be normal,* $\mu_i = 0$ *and player* j *is possibly committed* $0 < \mu_j \leq 1 - \frac{v_j}{v_i}$. *For any number* a_i *such that* $0 \leq a_i \leq \frac{1}{v_j}$, *a pair of functions*

$$\begin{aligned} \Pi_i(b) &= a_i b + (1 - a_i v_j) \\ \Pi_j(b) &= \frac{b}{v_i} \end{aligned}$$

for $0 \leq b \leq v_j$ *is an equilibrium. Similarly, for any number* $a_j \geq \frac{\mu_j}{v_i - v_j}$, *a pair of functions*

$$\begin{aligned} \Pi_i(b) &= \frac{b}{v_j} \\ \Pi_j(b) &= a_j b + (1 - a_j v_i) \end{aligned}$$

for $0 \leq b \leq v_j$ *is an equilibrium.*

If $a_j = \frac{\mu_j}{v_i - v_j}$, *then the equilibrium is the limit point of the sequence of unique equilibria in the two-sided reputation model (Myatt (2005)), when one considers a sequence of models with* $\mu_i \to 0$. *Notice that if we further assume that* $\mu_j \to 0$, *we have that* $\Pi_j(b) \to 1$ *for all* b *and hence the value-strong player wins the auction at the lowest price with probability one, regardless of what the relative reputational profile is.*

5 Behavioral types on two sides

This section investigates a scenario in which there is an uncertainty about the type of both players, $\mu_i > 0$ and $\mu_j > 0$. Abreu and Gul (2000) study a similar model in the context of bargaining.

It is not difficult to imagine the shape of the equilibrium. Once one of the players reduces his demand, he loses his reputation and the game enters the phase of one-sided reputation. According to Proposition 1, that player must keep reducing his demand until excess demand vanishes and the auction stops.

Assume that v is known and let $\omega_i = \frac{y_i}{x_i - y_i}$. Consider a strategy for bidder i, in which if a normal type ever decreases his demand from x_i, then he continues decreasing it until it reaches y_i. That is, after the initial reduction by player i, his strategy in all remaining stages $n < \bar{n}$ involves $F_i(b_i) = 1$ for any $b_i \geq p^n$.

In the initial stage $\bar{n}$, define $\Pi_i(b_i) = (1 - \mu_i) F_i(b_i)$ to be the probability that (unconditional) player i decreases his demand for the first time before or at the time when the price reaches b_i. The following proposition states that if functions (Π_1, Π_2) have a particular form then they generate a unique equilibrium payoff profile.

Proposition 2 *If* $0 < (\mu_i)^{\omega_i} \leq (\mu_j)^{\omega_j}$*, then the unique Perfect Bayesian equilibrium payoff profile is generated by a strategy profile that involves*

$$\begin{cases} \Pi_j(b) = 1 - \left(\frac{v-b}{v-r}\right)^{\omega_i} \\ \Pi_i(b) = 1 - \mu_i (\mu_j)^{\frac{\omega_j}{\omega_i}} \left(\frac{v-b}{v-r}\right)^{\omega_j} \end{cases}$$

for $b \in \left[r, v - (v - r)(\mu_j)^{\frac{1}{\omega_i}}\right]$.

The unique equilibrium payoff profile can be obtained by evaluating the expected payoff (defined in the Appendix, equation 8) at price r. Let $\gamma = \mu_i (\mu_j)^{-\frac{\omega_j}{\omega_i}}$, then in the two-sided version, payoffs ultimately are

$$\begin{cases} u_i(r) = y_i(v - r) \\ u_j(r) = (x_j(1-\gamma) + y_j\gamma)(v - r) \end{cases} \tag{1}$$

6 Conclusions

This paper discusses how to meaningfully analyze a model of multi-unit ascending auction with common values. The benchmark model, without perturbation, has many equilibria that are very different from each other and thus cannot provide any guidance to an auction designer. However, if this model is modified to allow a nonzero probability that there are players who insist on high quantities regardless of the price, then the unique equilibrium payoff emerges. The analysis of this equilibrium can now shed some light on the properties of this auction.

Auction designers usually ask two questions: "is the auction efficient?" and "what are the revenue properties of the auction?" Efficiency does not play any role in the context of common values as any division of units across agents is efficient.

The news for the revenue is mixed at best. The seller's revenue can be computed by taking the whole surplus available in the auction and subtracting the surplus that will be given to the bidders in equilibrium (equations 1). Let $0 < \mu_i \leq \mu_j$ and suppose that $x_i = x_j$. The revenue can approach the full surplus, vL, if the auction is symmetric and players are greedy (as $\mu_i \to 0$ for both players, $\frac{\mu_i}{\mu_j} \to 1$ and $x \to L$). But the revenue can approach as little as rL, if the auction is not symmetric or players are not greedy ($\frac{\mu_i}{\mu_j} \to 0$ or $x \to \frac{L}{2}$). Thus, the revenue depends sensitively on parameters that may not be easy to observe by the seller ex ante, such as $\frac{\mu_i}{\mu_j}$.

7 Appendix

7.1 Proof of Proposition 1

The following lemma provides an inductive step.

Lemma 1 *Fix $n = 0, 1, ..., X - L - 1$. Suppose that in any stage n with (p^n, z_1^n, z_2^n) in which excess demand is n and $\mu_j^n > 0$, $\mu_i^n = 0$, it is true that the only equilibrium market clearing price is p^n and equilibrium allocation to player i is y_i units.*

Then in any stage $n+1$ with $\left(p^{n+1}, z_1^{n+1}, z_2^{n+1}\right)$ and $\mu_j^{n+1} > 0$, $\mu_i^{n+1} = 0$, it is true that the only equilibrium market clearing price is p^{n+1} and equilibrium allocation to player i is y_i units.

Proof. Consider excess demand of $n + 1$ units at the starting price p^{n+1}. Fix an equilibrium.

Let $F_j(b|\theta_j)$ be the probability that normal player i decreases his demand for the first time before or at b, conditional on observing signal θ_j; let $\Pi_j(b|\theta_j) = \left(1 - \mu_j^{n+1}\right) F_j(b|\theta_j)$. Define $\tilde{R}_j(\theta_j)$ to be the greatest price at which player j of signal θ_j reduces his demand

$$\tilde{R}_j(\theta_j) = \min\left\{b : \Pi_j(b|\theta_j) = \lim_{b' \to \infty} \Pi_j(b'|\theta_j)\right\}$$

Let $R_j = \max_{\theta_j} \tilde{R}_j(\theta_j)$. Similarly define $\Pi_i(b|\theta_i)$, $\tilde{R}_i(\theta_i)$ and R_i. The implication of the assumption that $\mu_j > 0$ is that $\Pi_j(b|\theta_j) \leq 1 - \mu_j^{n+1}$ for all b and θ_j.

Notice that $R_i \leq R_j$ and $\Pi_i(R_i|\theta_i) = 1$ for all θ_i. This is because at any price higher than R_j normal player i believes with probability one that j is not going to reduce his demand.

Write R for R_i and contrary to the claim of the Lemma, suppose that $p^{n+1} < R$.[1]

For any $b \in (p^{n+1}, R)$, player j could reduce demand either at prices in the interval $[b, R]$ or at prices higher than R (including never). Let ζ_θ^b be the probability that player j reduces his demand at prices higher than R, conditional on price b and player j observing θ_j. That is

$$\zeta_\theta^b = \frac{1 - \Pi_j(R|\theta_j)}{1 - \Pi_j(b|\theta_j)}$$

Let $\zeta^b = \min_\theta \zeta_\theta^b$ and let $\left(w_{i\theta}^b, w_{j\theta}^b\right)$ be the expected payoff profile at b conditional on player j reducing his demand at prices less than R and conditional on signal profile being θ. Let $q_i^b(\theta_j|\theta_i)$ to be the posterior probability of θ_j conditional on b and θ_i.

1. It must be that for all θ_i

$$R < \lim_{b \to R} \sum_{\theta_j} q_i^b(\theta_j|\theta_i)\, v_\theta \tag{2}$$

[1] Assume that $\tilde{R}_j(\theta_j) \geq R$ and $\tilde{R}_i(\theta_i) = R$ for all $\theta \in \Theta$. This is without loss of generality. If there is θ_i' such that say $\tilde{R}_i(\theta_i') < R$, then only consider prices greater than $\tilde{R}_i(\theta_i')$ in the proof below and only these signals that still may exit after $\tilde{R}_i(\theta_i')$. Similarly for signal of player j.

Contrary to the statement, assume that there is θ_i such that

$$R = \lim_{b \to R} \sum_{\theta_j} q_i^b (\theta_j | \theta_i) v_\theta \tag{3}$$

Player θ_i compares the payoff from exiting at at price b and at a later price $b^{\#} \in (b, R)$. The minimal payoff that player θ_i gets from exiting at b is

$$\sum_{\theta_j} q_i^b (\theta_j | \theta_i) (v_\theta - b) y$$

The maximal payoff if player θ_i exits at $b^{\#}$ is no higher than

$$\sum_{\theta_j} q_i^b (\theta_j | \theta_i) \left[(v_\theta - b) x \left(\Pi_j \left(b^{\#} | \theta_j \right) - \Pi_j (b | \theta_j) \right) + \left(1 - \Pi_j \left(b^{\#} | \theta_j \right) \right) \left(v_\theta - b^{\#} \right) y \right]$$

which assumes that if j exits at a price in the interval $\left[b, b^{\#} \right)$, then the quantity x demanded by player i is fully satisfied at the lowest possible price b. The former cannot exceed the latter, because then player θ_i would exit before price reaches R. Hence

$$\sum_{\theta_j} q_i^b (\theta_j | \theta_i) \left[(v_\theta - b) x \left(\Pi_j \left(b^{\#} | \theta_j \right) - \Pi_j (b | \theta_j) \right) + \left(1 - \Pi_j \left(b^{\#} | \theta_j \right) \right) \left(v_\theta - b^{\#} \right) y \right] \geq \sum_{\theta_j} q_i^b (\theta_j | \theta_i) (v_\theta - b) y$$

or

$$\sum_{\theta_j'} q_i^b \left(\theta_j' | \theta_i \right) \left(1 - \Pi_j \left(b^{\#} | \theta_j' \right) \right) \sum_{\theta_j} q_i^{b^{\#}} (\theta_j | \theta_i) \left(v_\theta - b^{\#} \right) y \geq \sum_{\theta_j} q_i^b (\theta_j | \theta_i) (v_\theta - b) \left[y - x \left(\Pi_j \left(b^{\#} | \theta_j \right) - \Pi_j (b | \theta_j) \right) \right]$$

Now verify this inequality in the limit as $b^{\#}$ goes to R first and then b

goes to R. Let $b^{\#} \to R$. The LHS converges to zero, by 3. Hence

$$0 \geq \sum_{\theta_j} q_i^b (\theta_j|\theta_i) (v_\theta - b) \left[y - x \left(\lim_{b^{\#} \to R} \Pi_j \left(b^{\#}|\theta_j \right) - \Pi_j (b|\theta_j) \right) \right]$$

Consider the square bracket. Note that for every $\varepsilon > 0$, there is a price b^* strictly less than R such that for all $b > b^*$ and for all θ_j we have $\varepsilon > \lim_{b^{\#} \to R} \Pi_j \left(b^{\#}|\theta_j \right) - \Pi_j (b|\theta_j)$. Replace the square bracket with the term $y - x\varepsilon$, which is strictly lower. Choose $\varepsilon < \frac{y}{x}$, so that $y - x\varepsilon$ is strictly positive. Then we have that for every $b \in (b^*, R)$

$$0 > \sum_{\theta_j} q_i^b (\theta_j|\theta_i) (v_\theta - b)$$

This is a contradiction.

2. The bound ζ^b converges to one: $\lim_{b \to R} \zeta^b = 1$.

   Note that since $\Pi_j (b|\theta_j)$ is nondecreasing, ζ_θ^b is nondecreasing and bounded above, so it is converging. Suppose that there exists θ such that $\lim_{b \to R} \zeta_\theta^b < 1$; that is, player i of signal θ_i thinks that there is a mass point of players j who reduce demand at R. Then there is an $\epsilon > 0$ such that this θ_i reduces his demand at prices less than $R - \epsilon$. (Otherwise player θ_i, who is supposed to reduce demand at a price arbitrarily close to R, would benefit from delaying this reduction to a price just after R; the gain is bounded away from zero by (2), but the cost of delay is arbitrarily close to zero). This contradicts the definition of R.

3. Total payoff conditional on player j reducing his demand at some price higher than b cannot exceed the total surplus available at price b. Namely, for all θ

$$w_{i\theta}^b + w_{j\theta}^b \leq (v_\theta - b) (x + y) \tag{4}$$

4. With positive probability player i maintains his high demand for all prices less than R; this expected payoff has to be at least as high as

the expected payoff from reducing the demand immediately to y at a current price b. Namely, for all θ_i

$$\sum_{\theta_j} q_i^b(\theta_j|\theta_i)\left(\zeta_\theta^b(v_\theta - R)\,y + \left(1-\zeta_\theta^b\right)w_{i\theta}^b\right) \geq \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)\,y$$

Replace $w_{i\theta}^b$ by its upper bound implied by inequality (4). Multiply by the denominator of $q_i^\theta(\theta_j|\theta_i)$ and sum over all θ_i, to obtain one equation (where q_θ^b is the probability of θ, given b)

$$\sum_{\theta} q_\theta^b\left(\zeta_\theta^b(v_\theta - R)\,y + \left(1-\zeta_\theta^b\right)\left((v_\theta - b)(x+y) - w_{j\theta}^b\right)\right) \geq \sum_{\theta} q_\theta^b(v_\theta - b)\,y \tag{5}$$

Define Θ_L^b to be the set of θ for which

$$(v_\theta - R)\,y \leq (v_\theta - b)(x+y) - w_{j\theta}^b$$

and let Θ_U^b be its complement. Let $\lambda^b = \sum_{\Theta_U^b} q_\theta^b$.

Write the sums in (5) separately for Θ_L^b and Θ_U^b. Note that for all $\theta \in \Theta_L^b$, one can replace ζ_θ^b with its lower bound ζ^b and the inequality will hold. Likewise, for all $\theta \in \Theta_U^b$, one can replace ζ_θ^b with its upper bound 1.

$$\begin{aligned}\sum_{\Theta_L^b} q_\theta^b\left(\zeta^b(v_\theta - R)\,y + \left(1-\zeta^b\right)\left((v_\theta - b)(x+y) - w_{j\theta}^b\right)\right)& \\ + \sum_{\Theta_U^b} q_\theta^b(v_\theta - R)\,y& \\ \geq \sum_{\Theta_L^b} q_\theta^b(v_\theta - b)\,y + \sum_{\Theta_U^b} q_\theta^b(v_\theta - b)\,y&\end{aligned}$$

which can be written as

$$\begin{aligned}\sum_{\Theta_L^b} q_\theta^b\left(-\zeta^b(R-b)\,y + \left(1-\zeta^b\right)(v_\theta - b)\,x\right) - \left(1-\zeta^b\right)\sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b& \\ \geq (R-b)\,y\left(1-\lambda^b\right)&\end{aligned} \tag{6}$$

Notice that for any b, it must be that $\lambda^b > 0$, or in other words ζ^b occurs with positive probability. If not, then inequality (5) would be violated.

5. With positive probability player j reduces his demand at prices below R. The payoff from this cannot be lower than the payoff from maintaining the demand for x units. Namely, for all θ_j

$$\sum_{\theta_i} q_j^b(\theta_i|\theta_j)\, w_{j\theta}^b \geq \sum_{\theta_i} q_j^b(\theta_i|\theta_j)\,(v_\theta - R)\, x$$

Multiply this inequality by the denominator of $q_j^b(\theta_i|\theta_j)$ and sum over all θ_j. Then write the sums separately for Θ_L^b and Θ_U^b. Note that $w_{j\theta}^b \leq (v_\theta - b)\, x$; use this observation to replace $w_{j\theta}^b$ with its upper bound in the summation over Θ_U^b to obtain

$$\sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b \geq \sum_{\Theta_L^b} q_\theta^b (v_\theta - R)\, x - (R-b)\, x \left(1 - \lambda^b\right) \tag{7}$$

6. Replace $\sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b$ in inequality (6) with its lower bound in inequality (7)

$$\sum_{\Theta_L^b} q_\theta^b \left(-\zeta^b (R-b)\, y + \left(1-\zeta^b\right)(v_\theta - b)\, x\right)$$
$$- \left(1-\zeta^b\right)\left(\sum_{\Theta_L^b} q_\theta^b (v_\theta - R)\, x - (R-b)\, x\left(1-\lambda^b\right)\right)$$
$$\geq (R-b)\, y \left(1-\lambda^b\right)$$

Group all terms involving summation over Θ_L^b, divide by $R - b > 0$ and conclude that ζ^b is bounded away from 1 by a term involving only x and y

$$\zeta^b \leq \frac{x + y\lambda^b - y}{x + y\lambda^b} \leq \frac{x}{x+y} < 1$$

The implication of part 6 contradicts part 2. Thus $R = p^{n+1}$. Since it is impossible that player j reduced his demand at p^{n+1} it must be player i. ■

The hypothesis of this inductive step is trivially true for excess demand $n = 0$. The result follows.

7.2 Proof of the Proposition 2

Consider a stage when the reputation is two-sided and suppose $\left(\mu_j\right)^{\omega_j} \leq \left(\mu_i\right)^{\omega_i}$.

If a normal bidder i is the first to decrease his demand at price b_i, then the game moves to the next stage in which reputation is one-sided. His realized payoff is $y_i\left(v - b_i\right)$, by proposition 1. Similarly, if bidder j, decides to decrease his demand first, at some price b_j, then the game moves to the next stage in which reputation one-sided. Player i receives the quantity that he demanded in stage 0, namely x_i with a per unit profit $v - b_j$.

Recall that in general $\Pi_i\left(b_i\right)$ denotes a probability that player i decreases his demand by the time or at the time the price reaches b_i. Given the strategy of player j summarized in $\Pi_j\left(\cdot\right)$, the expected payoff of player i from exiting at price $b_i > r$ is

$$u_i\left(b_i\right) = x_i\left(v - r\right)\Pi_j\left(r\right) + \int_r^{b_i} x_i\left(v - b_j\right) d\Pi_j\left(b_j\right) + y_i\left(v - b_i\right)\left(1 - \Pi_j\left(b_i\right)\right) \tag{8}$$

Recall also the definition of R_j.

$$R_j = \min\left\{b : \Pi_j\left(b\right) = \lim_{b' \to \infty} \Pi_j\left(b'\right)\right\}$$

Let us establish a few properties of equilibria.

1. $R = R_1 = R_2 < v$.

2. For all $b > R$, $\Pi_1\left(b\right) = 1 - \mu_1$ and $\Pi_2\left(b\right) = 1 - \mu_2$. At price R, normal type of player i believes that player j will behave as if he was behavioral, as long as the prior $\mu_j > 0$. Hence, player i is not exiting only when he is behavioral himself.

3. If Π_j is discontinuous at any given point $b > r$, then there exists $\varepsilon > 0$ such that Π_i is constant on $(b - \varepsilon, b)$. If Π_j is discontinuous at any given point $b \geq r$ then Π_i is not discontinuous at b.

   Suppose that Π_j is discontinuous at b. Then normal player i who is supposed to exit before b but sufficiently close to b, is better off by waiting a tiny instant to some date just after b. The gain is discrete since a positive mass of players j exit at b and $x_i > y_i$, while the loss due to waiting is arbitrary small. For the second part suppose that Π_i is also discontinuous at b. Then there is positive probability that both bidders exit at b. On the other hand, if i waits a bit, this gives a discrete increase of a payoff, while a loss due to waiting is arbitrary small.

4. If Π_i is continuous at b then u_j is continuous at b.

   See the definition of u_j, above.

5. There is no interval $(b_1, b_2) \subseteq (r, R)$ such that both Π_i and Π_j are constant on (b_1, b_2). Both Π_i and Π_j are strictly increasing on (r, R).

   For the first claim suppose not, and let b_* be a supremum of upper bounds of all such intervals, so that at least one bidder exits with positive probability at every price just after b_*. Let i be a bidder, whose u_i is continuous at b_* (there is at least one). Note that u_i and u_j are both strictly decreasing on the interval (b_1, b_*), so for a fixed $b \in (b_1, b_*)$ there exists a positive constant ε such that for every $s \in (b_* - \varepsilon, b_* + \varepsilon)$ it is true that $u_i(b) > u_i(s)$. In particular, this means that bidder i cannot exit with positive probability at dates $s \in (b_*, b_* + \varepsilon)$ and Π_i is constant for all dates in $(b_1, b_* + \varepsilon)$. But then u_j is strictly decreasing on the interval $(b_1, b_* + \varepsilon)$, so bidder j cannot exit at these dates and Π_j is constant there too. This is a contradiction, because b_* was assumed to be the supremum of all such intervals. The second part is a by-product of the above.

6. Both Π_i and Π_j are continuous on (p, R).

Suppose Π_j has a jump at b, then Π_i is constant just before b. This contradicts that both are strictly increasing.

7. Both u_i and u_j are differentiable on (r, R).

   Utilities u_i and u_j are continuous on (r, R) because Π_i and Π_j are. Since Π_i and Π_j are strictly increasing u_i must be constant, hence differentiable.

8. It follows that $u_i'(b) = 0$ for all $b \in (r, R)$

The condition, $u_i'(b) = 0$ for $b \in (r, R)$, implies that

$$0 = (x_i - y_i)(v - b_i)\pi_j(b_i) - y_i(1 - \Pi_j(b_i))$$

$$\frac{1}{\omega_i}(v - b) = \frac{1 - \Pi_j(b)}{\pi_j(b)}$$

The general solution of the above condition is explicit and has the following form

$$\Pi_j(b) = 1 - (1 - \Pi_j(R))\left(\frac{v - b}{v - R}\right)^{\omega_i}$$

where the point $(R, \Pi_j(R))$ pins down the particular solution - the point yet unknown. Since $\omega_i > 0$ this function is increasing.

Two boundary conditions are $\Pi_j(R) = 1 - \mu_j$ and $\Pi_i(R) = 1 - \mu_i$, which implies

$$\begin{cases} \Pi_j(b) = 1 - \mu_j\left(\frac{v-b}{v-R}\right)^{\omega_i} \\ \Pi_i(b) = 1 - \mu_i\left(\frac{v-b}{v-R}\right)^{\omega_j} \end{cases} \tag{9}$$

The additional boundary conditions are $\Pi_i(r) \geq \Pi_j(r) = 0$, which imply

$$\begin{aligned} v - (v - r)\left(\mu_j\right)^{\frac{1}{\omega_i}} &= R \text{ and} \\ \left(\mu_j\right)^{\omega_j} &\leq (\mu_i)^{\omega_i} \end{aligned}$$

Eliminating R from the equations (9) gives the result:

$$\begin{aligned} &\Pi_j(b) = 1 - \left(\tfrac{v-b}{v-r}\right)^{\omega_i} \\ &\Pi_i(b) = 1 - \mu_i\left(\mu_j\right)^{\frac{\omega_j}{\omega_i}}\left(\tfrac{v-b}{v-r}\right)^{\omega_j} \end{aligned}$$

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